Can we compute fMRI brain activation directly from k-space?

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Abstract

In functional magnetic resonance imaging (fMRI), a process of determining statistically significant brain activation is commonly performed in terms of voxel time course measurements after image reconstruction. The image reconstruction and statistical activation processes are treated separately. In this manuscript, the relationship between complex-valued (Fourier) encoded k-space measurements and complex-valued image measurements from (Fourier) reconstructed images is described. The voxel time-series measurements are written in terms of spatio-temporal k-space measurements utilizing this spatial frequency k-space and image relationship. Voxel fMRI activation can be determined in image space for example using the Rowe-Logan complex-valued activation model [Rowe, D.B., and Logan, B.R. (2004). A complex way to compute fMRI activation. NeuroImage, 23 (3):1078-1092] in terms of the original k-space measurements. Additionally, the spatio-temporal covariance between reconstructed complex-valued voxel time series can be written in terms of the spatio-temporal covariance between complex-valued k-space measurements. Knowledge of the relationship between the spatio-temporal k-space measurements can be modeled in the more naturally acquired state rather than in a transformed state. This allows for the partitioning of the covariance matrix between the k-space measurements and hence voxel measurements into sources of covariation. Statistical associations between individual voxels or regions of interest can be quantified utilizing unmodeled sources of covariation.

1 Introduction

In functional magnetic resonance imaging (fMRI), the processes of image reconstruction (Kumar, et al., 1975; Haacke et al., 1999) and statistical activation (Bandettini et al., 1993; Friston et al., 1994) have been treated separately. The determination of statistically significant brain activation is in terms of voxel measurements after reconstruction. The relationship between the original k-space measurements and voxel measurements for each image is described. A permutation matrix is utilized to reorder the voxel measurements and statistical functional brain activation can be determined with complex-valued activation models (Nan and Nowak, 1999; Rowe and

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Logan, 2004,2005; Rowe, 2005a,b). A map of these activation statistics can then thresholded to determine statistically significant activation while adjusting for multiple compariations (Logan and Rowe, 2004).

2 Methods

In fMRI, data generally consist of two-dimensional slices acquired from an echo-planar imaging (EPI) pulse sequence. The $p_y \times p_x$ dimensional complex-valued spatial frequency measurement S_C of a slice consists of a $p_y \times p_x$ dimensional matrix of true underlying noiseless complex-valued spatial frequencies S_{0C} and a $p_y \times p_x$ dimensional matrix of complex-valued measurement error E_C that can be represented as

$$S_C = (S_{0R} + iS_{0I}) + (E_R + iE_I) \tag{2.1}$$

where i is the imaginary unit while S_{0R}, S_{0I}, E_R and E_I are real and imaginary matrix valued parts of the true spatial frequencies and measurement noise. Let Ω_{Cx} and Ω_{Cy} be $p_x \times p_x$ and $p_y \times p_y$ complex-valued Fourier matrices such that

$$\Omega_{Cy} = \Omega_{Ry} + i\ \Omega_{Iy} \quad \text{and} \quad \Omega_{Cx} = \Omega_{Rx} + i\ \Omega_{Ix} \tag{2.2}$$

where Ω_{Ry} and Ω_{Rx} are real while Ω_{Iy} and Ω_{Ix} are imaginary matrix valued parts. Then, the $p_y \times p_x$ complex-valued inverse Fourier transformation reconstructed image R_C of S_C can be written as

$$\begin{aligned} R_C &= \Omega_{Cy} * S_C * \Omega_{Cx}^T \\ &= \Omega_{Cy}(S_{0R} + iS_{0I})\Omega_{Cx}^T + \Omega_{Cy}(E_R + iE_I)\Omega_{Cx}^T \\ &= R_{0C} + N_C \end{aligned}$$

where R_C has a true mean R_{0C} and measurement error N_C while "T" denotes transposition. If Ω_{Cx} is a Fourier matrix, it is $[\Omega_{Cx}]_{jk} = \kappa \left(\omega^{jk}\right)$ where $\kappa = 1$ and $\omega = \exp[-i2\pi(j-1)(k-1)/p_x]$ for the forward transformation while $\kappa = 1/p_x$ and $\omega = \exp[+i2\pi(j-1)(k-1)/p_x]$ for the inverse transformation, where $j, k = 1, ..., p_x$. The complex-valued matrices for reconstruction Ω_x and Ω_y need not be exactly Fourier matrices but may be Fourier matrices that include adjustments for magnetic field inhomogeneities derieved from phase maps or reconstruction matrices for other encoding procedures.

This inverse Fourier transformation image reconstruction process can be equivalently described as the pre-multiplication of the complex-valued spatial frequencies in the form of a real-valued vector s by a real-valued matrix representation Ω of the complex-valued Fourier matrices

$$\begin{aligned} \left(r \right) &= \left(\Omega \right) * \left(s \right) \\ \begin{pmatrix} r_R \\ r_I \end{pmatrix} &= \begin{pmatrix} \Omega_R & -\Omega_I \\ \Omega_I & \Omega_R \end{pmatrix} * \begin{pmatrix} s_R \\ s_I \end{pmatrix} \end{aligned} \tag{2.3}$$

where the real-valued representation r that is of dimension $2p_xp_y \times 1$ of the complex-valued image has a true mean and measurement error. The real-valued vector of spatial frequencies is formed by

$$s = \text{vec}(S_R^T, S_I^T)$$

where S_R and S_I denote the real and imaginary parts of S_C and vec$(\cdot)$ denotes the vectorization operator that stacks the columns of its matrix argument. In addition, the matrix elements of Ω are

$$\Omega_R = [(\Omega_{yR} \otimes \Omega_{xR}) - (\Omega_{yI} \otimes \Omega_{xI})] \quad (2.4)$$

$$\Omega_I = [(\Omega_{yR} \otimes \Omega_{xI}) + (\Omega_{yI} \otimes \Omega_{xR})] \quad (2.5)$$

where $\otimes$ denotes the Kronecker product that multiplies every element of its first matrix argument by its entire second matrix argument. If the mean and covariance of the spatial frequency measurement vector s that is of dimension $2p_xp_y \times 1$ are s_0 and Γ, then the mean and covariance of the reconstructed voxel measurements r are Ωs_0 and $\Omega\Gamma\Omega^T$.

In fMRI, a series of the previously described slices are acquired. Denote the $p_y \times p_x$ random complex-valued spatial frequency matrix at time t as $S_{Ct} = S_{0Ct} + E_{Ct}$ and define $s_t = \text{vec}(S_{Rt}^T, S_{It}^T)$, where S_{Rt} and S_{It} are the real and imaginary parts of S_{Ct} for time points $t = 1, ..., n$. Define the total number of voxels in the image, which is the same as the number of complex-valued k-space measurements to be $p = p_xp_y$. This sequence of measured spatial frequency vectors can be collected into a $2p \times n$ matrix $S = (s_1, ..., s_n)$ where the t^{th} column contains the p real k-space measurements stacked upon the p imaginary k-space measurements for time t. Having done this, n reconstructed images can be formed by the $2p \times n$ matrix $R = \Omega S$ where the t^{th} column of R contains the p real voxel measurements stacked upon the p imaginary voxel measurements for time t, $t = 1, ..., n$.

The k-space measurements and the image voxel measurements can be stacked as $s = \text{vec}(S)$ and $r = \text{vec}(R)$. Note that s and r and have been redefined from their previous definition. If the mean and covariance of the $2np \times 1$ vector of spatial frequency measurements s are s

2005) can be found by choosing the phase $_{jt} = u_t^T$ where u_t is the t^{th} row of a phase design matrix and are phase regression coe cients.

This can be rearranged and written with $= \text{vec}(\ )$ as

$$\begin{pmatrix} _{R1} \\ _{I1} \\ \vdots \\ _{Rp} \\ _{Ip} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} X & 0 \\ 0 & SX \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} X_p & 0 \\ 0 & SX_p \end{pmatrix} \end{pmatrix} \begin{pmatrix} _1 \\ _1 \\ \vdots \\ _p \\ _p \end{pmatrix} + \begin{pmatrix} \epsilon_{R1} \\ \epsilon_{I1} \\ \vdots \\ \epsilon_{Rp} \\ \epsilon_{Ip} \end{pmatrix} \tag{2.7}$$

where $= (^T_{R1}, ^T_{I1}, ..., ^T_{Rp}, ^T_{Ip})^T$ is a vector containing the real and imaginary reconstructed voxel measurements and $\epsilon = (\epsilon^T_{R1}, \epsilon^T_{I1}, ..., \epsilon^T_{Rp}, \epsilon^T_{Ip})^T$ is a vector containing the real and imaginary errors of the reconstructed voxel measurements. The model can simply be written as $= \mu + \epsilon$. For example, with constant phase model, the mean is $\mu = (_{2p} \otimes X)[(\cos\ _1, \sin\ _1) \otimes\ ^T_1, ..., (\cos\ _p, \sin\ _p) \otimes\ ^T_p]^T$.

The rearrangement of the voxel measurements from r to is a linear transformation and can be achieved through a permutation matrix (described in Appendix A) to form $= r$. In terms of the original k-space measurements the voxel time courses are $= (_n \otimes \Omega)\text{vec}(\text{vec}(S^T_{R1}, S^T_{I1}), ..., \text{vec}(S^T_{Rn}, S^T_{In}))$. A permutation matrix is a square matrix that can be obtained by permuting (rearranging) either the columns or rows of an identity matrix (Harville, 1999). A permutation matrix is of full rank and therefore nonsingular and also invertible. Having done this linear transformation, the mean and covariance of are $\mu = (_n \otimes \Omega)s_0$ and $\Lambda = (_n \otimes \Omega)\Delta(_n \otimes \Omega^T)\ ^T$. Since the matrices Ω and that convert k-space measurements s to voxel measurements are known *a priori*, the expression $= (_n \otimes \Omega)s$ can be inverted to write $s = (_n \otimes \Omega^{-1})\ ^{-1}$ in terms of the parameters as

$$\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = \underbrace{\ ^{-1}\left[_n \otimes \Omega^{-1}(_2 \otimes X)\right]} \begin{pmatrix} _1 \cos\ _1 \\ _1 \sin\ _1 \\ \vdots \\ _p \cos\ _p \\ _p \sin\ _p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} \tag{2.8}$$

then the optimization for the regression coe cients ( ) and phases ( ) can be performed in k-spacesto yield the same parameter estimates. Activations can then be computed from Rowe's complex activation models.

Using ordinary least squares or a normal distributional specification on the errors, the voxel-wise regression coe cients and phases can be determined to yield the same point estimators as in Logan and Rowe (2004). The Rowe-Logan unconstrained alternative hypothesis estimators (with hats) for $_1: \neq 0$ along with the constrained null hypothesis estimators (with tildes) for $_0: = 0$ in voxel j are

$$\begin{aligned} \hat{\ }_j &= \tfrac{1}{2}\tan^{-1}\left[\frac{\beta^T\ X'X\ \beta}{\beta^T\ X'X\ \beta\ -\beta^T\ X'X\ \beta\ /2}\right] & \tilde{\ }_j &= \tfrac{1}{2}\tan^{-1}\left[\frac{\beta^T\ X'X\ \beta}{\beta^T\ X'X\ \beta\ -\beta^T\ X'X\ \beta\ /2}\right] \\ \hat{\ }_j &= \hat{\ }_{Rj}\cos\hat{\ }_j + \hat{\ }_{Ij}\sin\hat{\ }_j & \tilde{\ }_j &= \Psi\left[\hat{\ }_{Rj}\cos\tilde{\ }_j + \hat{\ }_{Ij}\sin\tilde{\ }_j\right] \end{aligned} \tag{2.9}$$

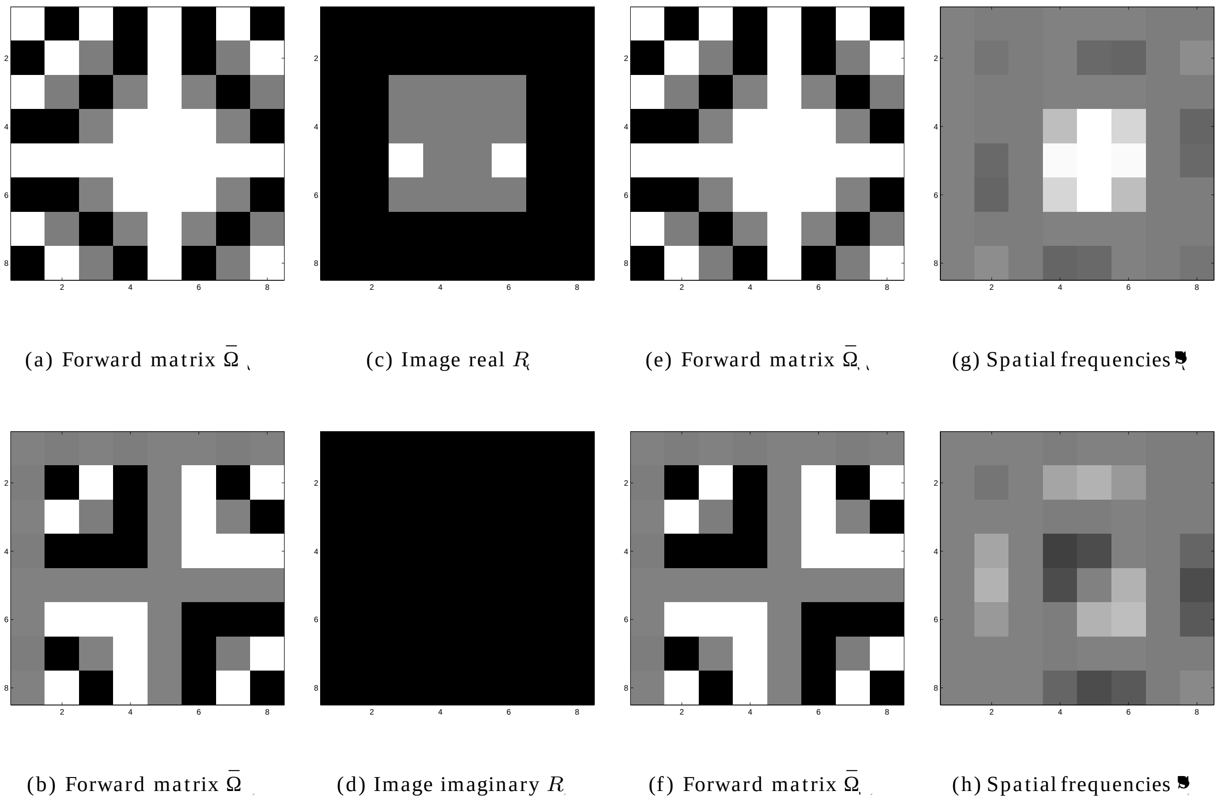
where is an $r\times(+1)$ matrix of full row rank, $\Psi = \ _q \ _1 - (\ )^{-1} \ '[(\ )^{-1} \ ']^{-1}$, $\hat{}_{Rj} = (\ )^{-1} \ ' \ _{Rj}$, and $\hat{}_{Ij} = (\ )^{-1} \ ' \ _{Ij}$, while $_{Rj}$ and $_{Ij}$ are the $n\times 1$ vectors of real and imaginary voxel observations.

The variances and covariances for example with a specification of uncorrelated temporal k-space measurement vectors (s_t) yields the covaraince matrix $\Lambda = \ _n \otimes \Omega\Gamma\Omega^T$ for the voxel measurements. Define the voxel measurement covariance matrix to be $\Sigma = \Omega\Gamma\Omega^T$. Having estimated the voxel-wise regression coe cients and phases, we can estimate the mean of the vector of voxel measurements by $\hat{\mu}_l$ (under the alternative hypothesis) and the mean of the matrix of voxel measurements R by $\hat{} = \overline{\text{vec}}(\ ^{-1}\hat{\mu}_l)$. Here $\overline{\text{vec}}(\cdot)$ is the operator that is the inverse operation of the vec$(\cdot)$ operator. The voxel covariance matrix is

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix} \tag{2.10}$$

where the partitioned matrix elements are

$$\Sigma_{11} = \Omega$$

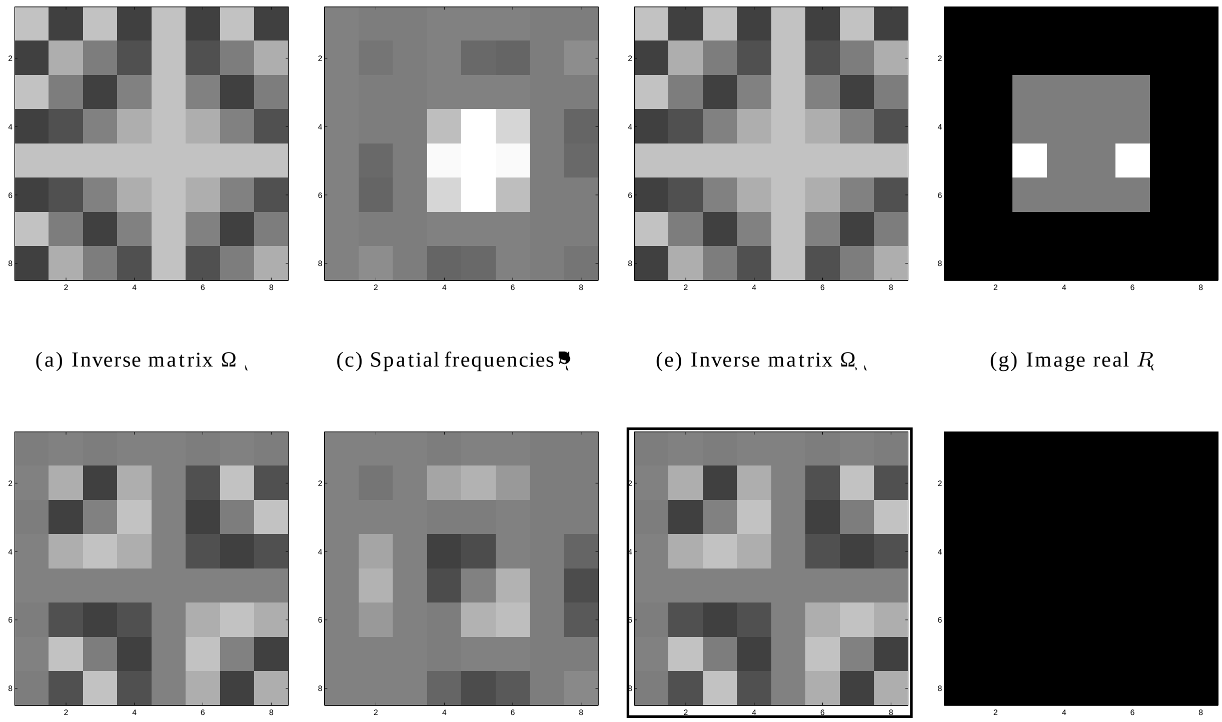


(a) Forward matrix $\bar{\Omega}$ (c) Image real R (e) Forward matrix $\bar{\Omega}$ (g) Spatial frequencies

(b) Forward matrix $\bar{\Omega}$ (d) Image imaginary R (f) Forward matrix $\bar{\Omega}$ (h) Spatial frequencies

Figure 1: Complex-valued 2D forward Fourier transform

3 Example



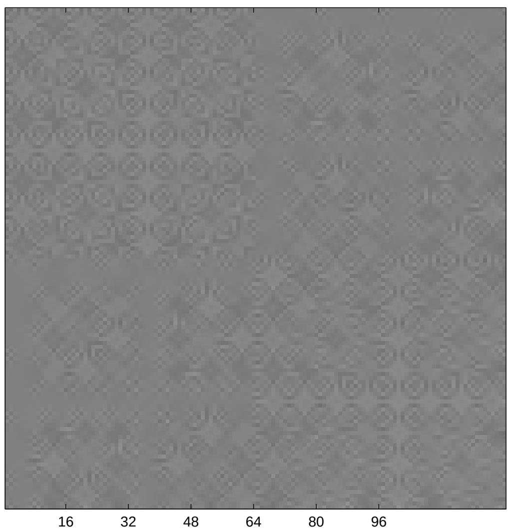
(a) Inverse matrix Ω (c) Spatial frequencies (e) Inverse matrix Ω (g) Image real R

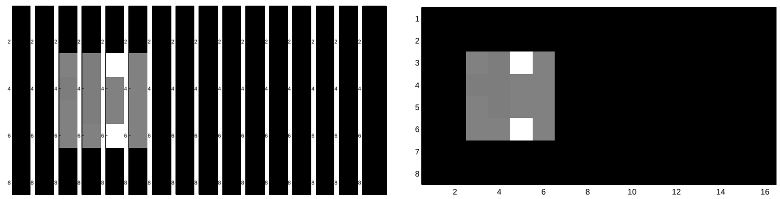
(b) Inverse matrix Ω (d) Spatial frequencies (f) Inverse matrix Ω (h) Image imaginary R

Figure 2: Complex-valued 2D inverse Fourier transform

imaginary image part S_I in Figure 2d by the complex-valued inverse Fourier matrix Ω_{Cy} presented as an image with real part Ω_{Ry} in Figure 2a and imaginary part Ω_{Iy} in Figure 2b then post-multiply the result by the transpose of the symmetric inverse Fourier matrix Ω_{Cx} presented as an image with real part Ω_{Rx} in Figure 2e and imaginary part Ω_{Ix} in Figure 2f. The recovered complex-valued image R_C is presented with real part R_R in Figure 2g and imaginary part R_I in Figure 2h.

real-valued representation often called an isomorphism in mathematics. To use this representation, join the transpose of the real and imaginary parts of the spatial frequency (k-space) values given in Figure 2c and Figure 2d respectively that are of dimension $p_y \times p_x$ into a single real-valued matrix $S^T = (S_R^T, S_I^T)$ that is of dimension $p_x \times 2p_y$ as in Figure 3a. Then stack the columns of S^T as shown partitioned in Figure 3b into a single vector $s = \text{vec}(S_R^T, S_I^T)$ as presented in Figure 4b. This gives us a real-valued vector representation of the matrix of spatial frequency (k-space) values.





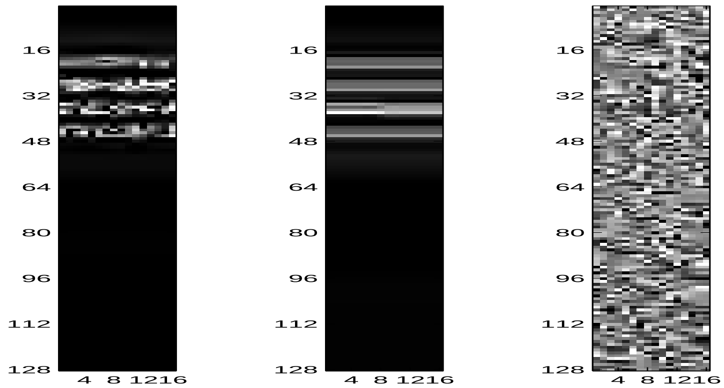
(a) partitioned images R

(b) combined image R = (R, R)

Figure 5: Vector to matrix image values.

vector form of the spatial frequencies for an image similar to that in Figure 4b.

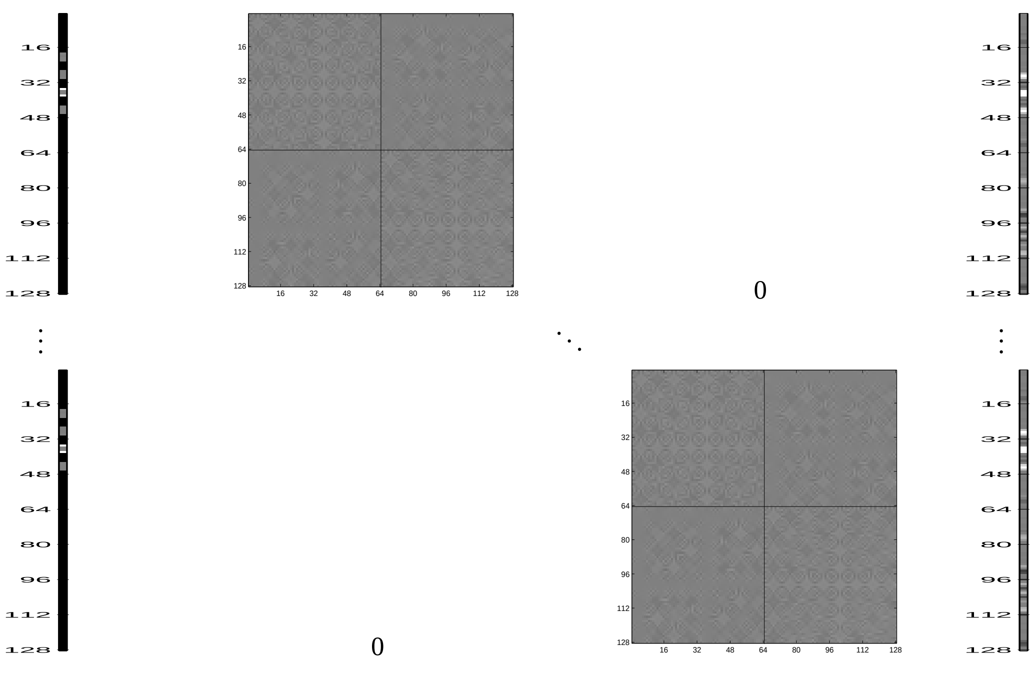
The mean on images contained voxels with values $_{0}$



(a) noisy $R = \Omega$ (b) true $R_0 = \Omega\,{}_0$ (c) error $= \Omega$

Figure 7: Reconstructed noisy images.

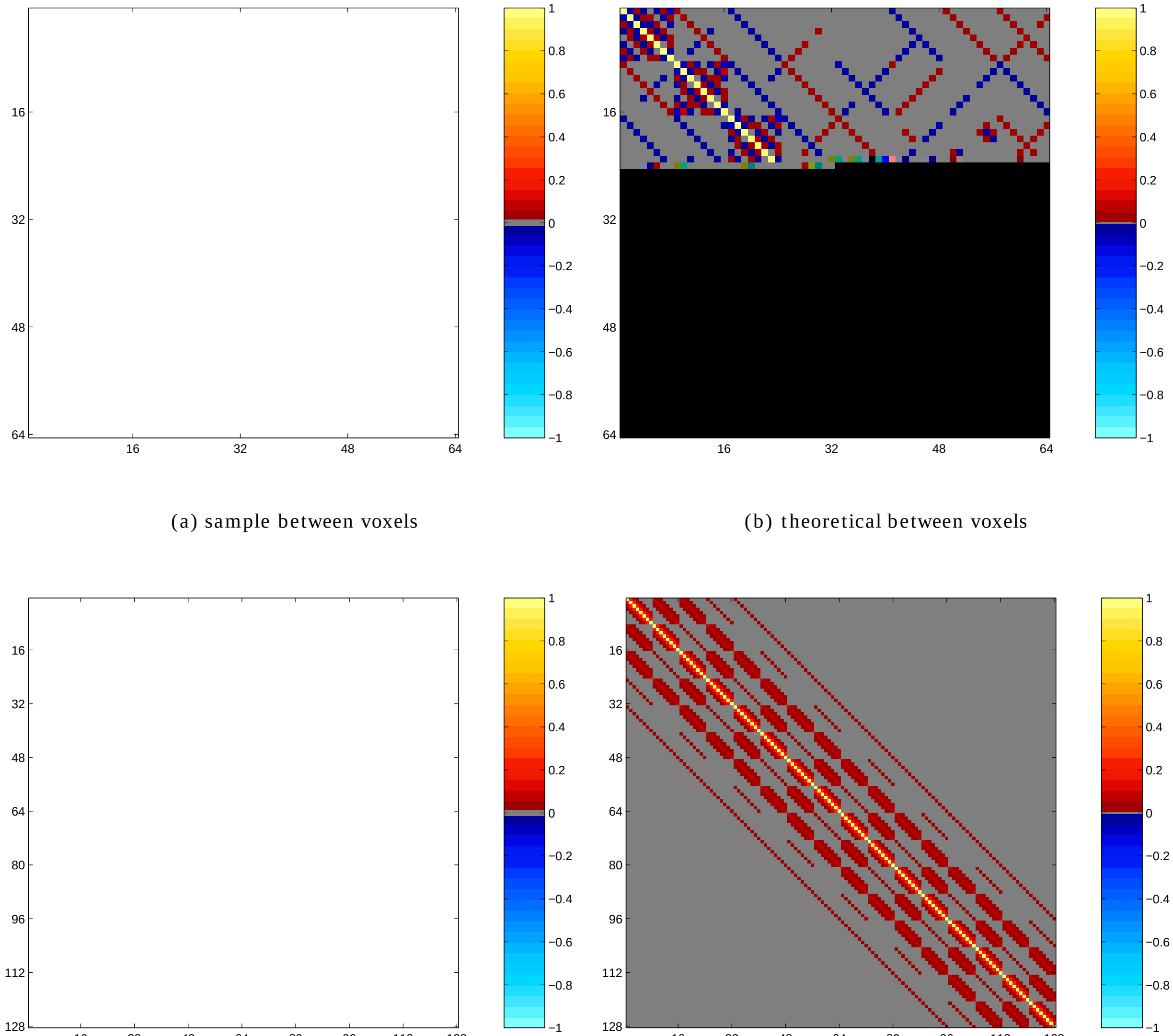
of $_2 = .5$ while the $p_x \times p_x$ correlation matrix Γ is taken to be an AR(1) correlation matrix with $(i, j)^{th}$ element $^{|\ -j|}$ where $= 0.5$.



(a) noisy (b) reconstruction matrix ($\otimes\, \Omega$) (c) noisy

Figure 8: Reconstructed vectorized noisy images.

Each matrix image in Figure 6a, b, and c was pre-multiplied by the (inverse Fourier transform) image



(a) sample between voxels

(b) theoretical between voxels

(c) sample between frequencies

(d) theoretical between frequencies

Figure 11: Correlation matrices

permutation matrix that form the n imaginary measurements within the first second have a 1 in column $t = p + 2, 3p + 2, 5p + 2, ..., 2(n-1)p + p + 2$. This general pattern continues so that the t^{th} row within the $(2p-1)^{th}$ set of n rows of the permutation matrix that form the n real measurements within the p^{th} voxel have a 1 in column $t = 0p + p, 2p + p, 4p + p, ..., 2(n-1)p + p$. The t^{th} row within the second set of n rows of the permutation matrix that form the n imaginary measurements within the first second have a 1 in column $t = p + p, 3p$ pt

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